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Irreducible Graphs—Part 2*

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This paper gives the complete list of six cubic graphs which are irreducibly non-representable on the projective plane.

INTRODUCTION

This paper continues a previous paper Milgram [12]. Results are numbered, and terms and notations are used to conform with that paper. The references are also repeated.

8. SIX CUBIC GRAPHS

We will now go into detail concerning some specific cubic graphs. First we will list all cubic graphs with extra θ irreducibly non-representable on the projective plane and then we will present two additional cubic graphs without extra θ which are irreducibly non-representable on the projective plane.

8.1. *Non-representability on the Projective Plane*

LEMMA 8.1.1. *There is only one disconnected cubic graph irreducibly non-representable on the projective plane.*

Proof. Each component must be irreducibly non-planar by Lemma 3.1. When an edge is removed from any component, that component must have a planar representation and can be represented in one of the 2-cells

* The results in this paper were first presented in a talk at Rockefeller University in February, 1967.

bounded by the other copy of $K_{3,3}$ on the projective plane. We will call this graph I_1.

We recall from Lemma 3.2 that there are no 1-connected graphs irreducibly non-representable on the projective plane.

LEMMA 8.1.2. *There is at most one 2-connected cubic graph irreducibly non-representable on the projective plane.*

Proof. If there were a 2-connected cubic graph irreducibly non-representable on the projective plane, then we can find two edges e_1 and e_2 in such a graph which disconnect the graph. Let the graph be $G = G_1 + G_2 + e_1 + e_2$ where G_1 and G_2 are disconnected components of G. If either $M_1 = G_1 + e_1 + e_2$ or $M_2 = G_2 + e_1 + e_2$ were outside graphs, then G would itself be representable on the projective plane. For, if both were outside, G would be planar. On the other hand, if M_1 were outside and M_2 were not, then by Lemma 5.1 M_1 would not have a θ and, since it has only two nodes of degree one, M_1 would be a single edge, which is contrary to assumption. But if both M_1 and M_2 are non-outside, they each have the minimal outside graph with two free edges as subgraphs; hence G has the graph I_2 as a subgraph. The graph I_2 is with extra θ and Lemma 5.2 shows it to be non-representable on the projective plane.

LEMMA 8.1.3. *There is at most one 3-connected cubic graph irreducibly non-representable on the projective plane.*

Proof. Consider such a graph G. We can find three edges e_1, e_2, and e_3 which disconnect the graph into components G_1 and G_2. Let $M_1 = G_1 + e_1 + e_2 + e_3$ and let $M_2 = G_2 + e_1 + e_2 + e_3$. Then both M_1 and M_2 are non-outside. For, if both were outside, G would be planar. If M_1 were outside and M_2 were not, then M_1 would have no θ. Since M_1 has three free edges, but no three circuit by Lemma 3.3, it would have to consist of three concurrent edges only, contrary to hypothesis. Thus M_1 and M_2 are both nonoutside. Then each has a subgraph homeomorphic to the minimal non-outside graph with three free edges, or G has a subgraph of the form I_3. By Lemma 5.2, I_3 is not representable on the projective plane. This graph is, of course, with extra θ.

LEMMA 8.1.4. *There is no cubic graph irreducibly non-representable on the projective plane with a θ disjoint from a $K_{3,3}$.*

Proof. Assume the contrary and let G be such a graph. Then $G - G_K$ contains a non-outside subgraph or it would be representable by the construction in the proof of Lemma 5.1. If $G - G_K$ contains the minimal non-outside graph with two free edges, H_2, then $G = H_2 + G_K$ because $H_2 + G_K$ is already non-representable. But this contains I_2 as a proper subgraph, which is contrary to the hypothesis that G is irreducible. Similarly, if $G = H_3 + G_K$ (recall that H_3 is the minimal non-outside graph with three free edges), then G would have I_3 as a proper subgraph.

LEMMA 8.1.5. *All connected cubic graphs with extra θ irreducibly non-representable on the projective plane are formed by selecting $M_j = H_i$, $j = 1,2$, $i = 1,2$, and joining the free edges of M_1 to parts of M_2 and vice versa.*

Proof. Select M_1 as disjoint from G_{Th}. M_1 has one of the H's as a subgraph by Lemma 5.1. M_1 cannot be disjoint from a G_K by Lemma 8.1.4. But, if we add free edges to the nodes of degree two of $G - M_1$, we have an H.

This last result (Lemma 8.1.5) enables us to obtain the last possible cubic graph with extra θ which is irreducibly non-representable on the projective plane. This graph is I_4. The reader can try all possibilities suggested by Lemma 8.1.5. In addition to the graphs I_1 to I_4 which comprise all candidates for graphs with extra θ, we wish to examine two graphs without extra θ. To show that these additional two graphs are non-representable on the projective plane we will need some computational aids. We recall that there is only one representation of $K_{3,3}$ on the projective plane (Lemma 7.1).

A *Hamilton circuit* of a graph G is a circuit which contains all the nodes of G and each edge of the circuit appears once only.

LEMMA 8.1.6. *For every Hamilton circuit, C, of $K_{3,3}$, there is a representation of $K_{3,3}$ on the projective plane such that C bounds a 2-cell.*

Proof. Each Hamilton circuit in $K_{3,3}$ corresponds to a permutation of the nodes of $K_{3,3}$. In $K_{3,3}$, this permutation is an automorphism. Thus, the single representation of $K_{3,3}$ on the projective plane, to which all others are isomorphic, gives the result.

LEMMA 8.1.7. *If the nine edges of $K_{3,3}$, numbered* 1 *to* 9, *are arranged in the* 3×3 *matrix*:

$$\begin{matrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{matrix}$$

where the rows and columns give adjacencies, then the rows and columns of the magic square:

2	9	4
7	5	3
6	1	8

gives the six sets of three mutually non-adjacent edges.

LEMMA 8.1.8. *There are six Hamilton circuits of* $K_{3,3}$.

Proof. Each set of three mutually non-adjacent edges, s, has $K_{3,3} - s$ as a Hamilton circuit. Using Lemma 8.1.7 there are six of these.

LEMMA 8.1.8. *The six Hamilton circuits of* $K_{3,3}$:

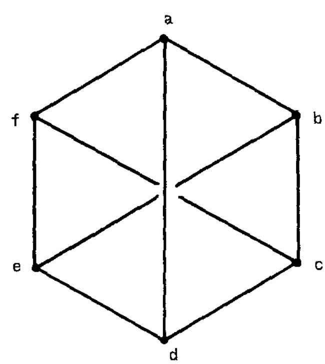


FIGURE 8.1.8.1

are given by columns of vertices:

a	1	1	1	1	1	1
b	2	2	2	2	4	4
c	3	5	3	5	3	5
d	4	4	6	6	2	2
e	5	3	5	3	5	3
f	6	6	4	4	6	6

A *simplest* imbedding of a graph, G, in a surface, S, is a 2-cell imbedding of G in a surface S of the smallest Euler characteristic (see [7, page 305]).

LEMMA 8.1.9. *Let G' be a cubic graph which has a simplest imbedding in S. Assume that there are three nodes of degree two, n_1, n_2, n_3 for which there is only one path p in G' going through n_1, n_2, and n_3* in that order *such that p is part of the boundary of a 2-cell imbedding of G' in S. Then, for a node n_4 of degree two of G' and not on p, the graph*

$$G = G' + (n_1, n_5) + (n_5, n_3) + (n_2, n_6) + (n_5, n_6) + (n_6, n_4)$$

is not representable on S.

Proof. We have added two nodes n_5 and n_6 and five edges to G'. If G were representable, n_1, n_2, n_3, and n_4 must appear on a circuit C which is represented in a 2-cell in the imbedding restricted to G', since the representation restricted to G' must be a 2-cell imbedding. But the graph formed by C and the nodes n_5 and n_6 (Figure 8.1.9.1) and their

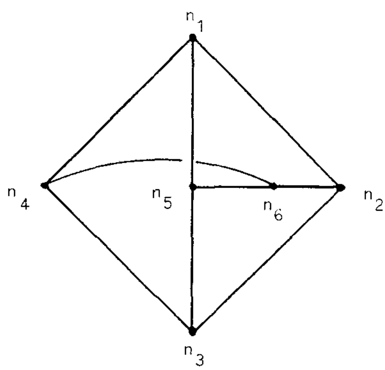


FIGURE 8.1.9.1

attached edges is a $K_{3,3}$, which cannot be represented in a 2-cell with boundary.

LEMMA 8.1.10. *Let e_2 be an edge of $K_{3,3}$ and V_1 and V_3 be its adjacent nodes. Let e_1 be an edge of $K_{3,3}$ adjacent to e_2 at V_1 and e_3 be an edge of $K_{3,3}$ adjacent to e_2 at V_3. Let K' be the graph formed from $K_{3,3}$ by adding nodes n_1, n_2, and n_3 of degree two to the edges e_1, e_2, and e_3 of $K_{3,3}$, respectively. Then in any representation of K' on the projective plane such that n_1, n_2, n_3 touch the same region R there is only one path p in K' that starts at n_1, goes through n_2, and ends in n_3 such that p is part of the boundary of R.*

Proof. $K_{3,3}$ is a divider of the projective plane. Thus, for e_1, e_2, and e_3 to touch one region, R, the other edges at V_1 and V_3 cannot touch R. Thus, the only possible path from n_1 to n_2 to n_3 is along e_1 to e_3.

LEMMA 8.1.11. *The two cubic graphs without extra θ listed below (I_5 and I_6) are non-representable on the projective plane.*

Proof. Each is of the form in Figure 8.1.11.1, which by Lemmas 8.1.9 and 8.1.10 ensures that they are non-representable. It is our object to

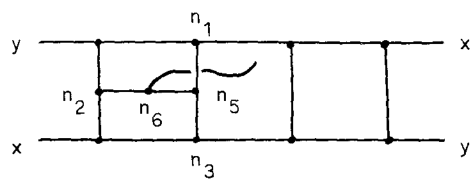


FIGURE 8.1.11.1

show that every cubic graph without extra θ irreducibly non-representable on the projective plane must contain the configuration above.

We have established the fact that the six graphs are not representable on the projective plane.

8.2. *Irreducibility of the Six Graphs*

We intend to show that I_1 through I_6 are irreducible and that they are not isomorphic in pairs. We have already shown I_1 to be irreducible. We will actually give a representation on the projective plane for each graph with one edge removed. In order to do this task without an electronic computer, we use some simple computational aids.

An *automorphism* of a graph G is a permutation g of the nodes such that, for any two nodes m and n, m is adjacent to n if and only if $g(m)$ is adjacent to $g(n)$.

Automorphisms may be extended to edges of a graph since the edges define adjacencies of nodes.

A *transitivity class* of edges of a graph G is a set of edges such that, for any two edges x and y in that set, there is an automorphism g of G such that $x = g(y)$.

In what follows, we will demonstrate the irreducibility of these six graphs. In doing so, we will also demonstrate that all graphs are not isomorphic in pairs. Our basic technique will be to compute transitivity classes of edges. Irreducibility will be demonstrated by removing a representative edge of a transitivity class and showing that the resulting graph is representable on the projective plane.

Our plan for the computation of transitivity classes is straightforward. We will give a table of permutations of the nodes for each graph. These permutations will be automorphisms. For example, the first automorphism given for I_5 is illustrated in Figure 8.2.1. These automorphisms will enable us to compute transitivity classes.

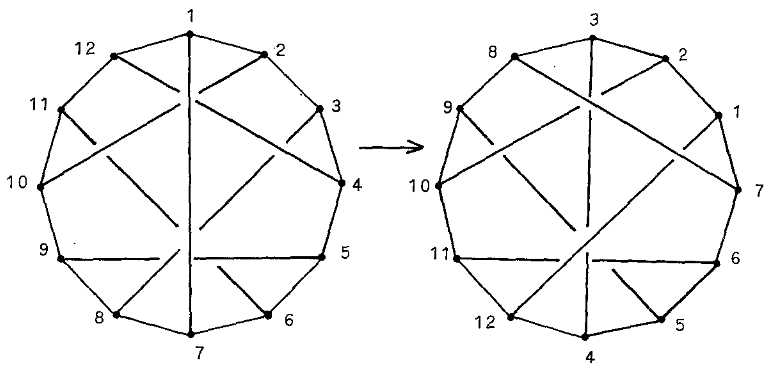


FIGURE 8.2.1

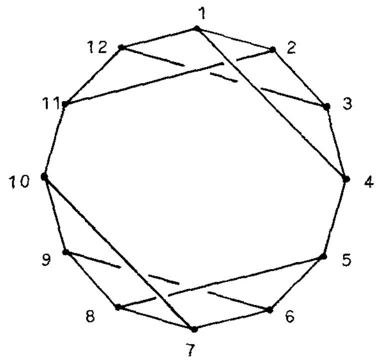
A *reduced* graph G' of an irreducible graph G is any graph $G' = G - e$ where e is an edge of G. It is clear that:

LEMMA 8.2.1. *Let G' be a reduced graph of G. If G' is homeomorphic to no reduced graph of H, then H and G are not homeomorphic.*

The proof of irreducibility of each graph then is a simple matter of finding a representation for each subgraph formed by deleting one edge of a transitivity class. The proof that the six graphs are not isomorphic in pairs is simplified by the examination of these reduced subgraphs.

Remark. It is not necessarily true that the automorphisms that we give actually generate the group of automorphisms (e.g., the group of automorphisms of I_6 is certainly larger than the cyclic group of order 8 that is generated by the single automorphism given). All that is necessary for our purpose is that the subgroup generated by the automorphisms given do indeed have the transitivity classes which are presented. The proof that the transitivity classes given are distinct is by examination of the reduced graphs. If two reduced graphs $G' = G - e'$ and $G'' = G - e''$ of G are distinct then e' and e'' must be in two distinct transitivity classes.

THE GRAPH I_2

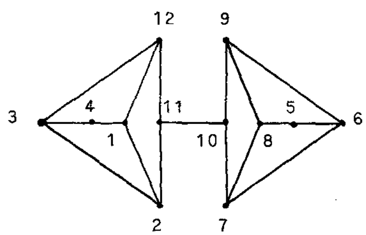


Transitivity Classes

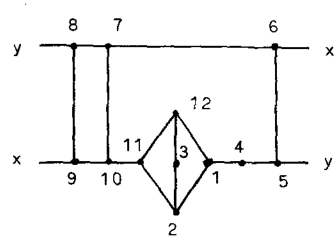
(4, 5)	(3, 4)	(1, 2)
(10, 11)	(11, 12)	(2, 3)
	(1, 4)	(1, 12)
	(9, 10)	(7, 8)
	(5, 6)	(3, 12)
	(7, 10)	(6, 7)
	(2, 11)	(8, 9)
	(5, 8)	(6, 9)

Automorphisms

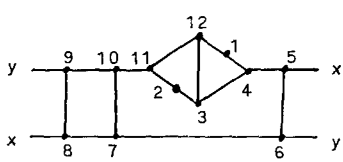
1	2	3	4	5	6	7	8	9	10	11	12
2	1	12	11	10	9	8	7	6	5	4	3
2	3	12	11	10	7	6	9	8	5	4	1
8	7	6	5	4	3	2	1	12	11	10	9



I_2 - (4,5)



I_2 - (3,4)

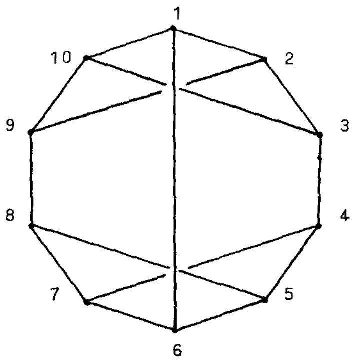


I_2 - (1,2)

THE GRAPH I_3

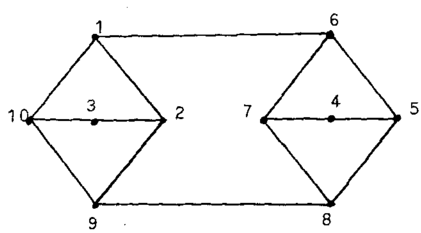
Transitivity Classes

(3, 4)	(1, 2)
(1, 6)	(2, 3)
(8, 9)	(1, 10)
	(5, 6)
	(4, 5)
	(6, 7)
	(4, 7)
	(1, 10)
	(3, 10)
	(2, 9)
	(5, 8)
	(7, 8)
	(9, 10)

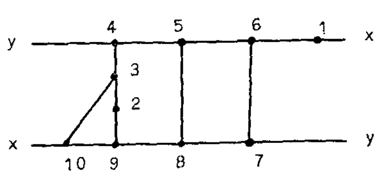


Automorphisms

1	2	3	4	5	6	7	8	9	10
3	2	1	6	5	4	7	8	9	10
1	10	9	8	7	6	5	4	3	2
6	5	4	3	2	1	10	9	8	7

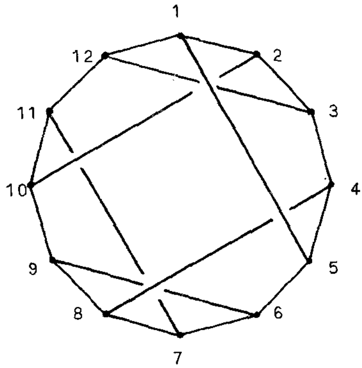


$I_3 - (3,4)$



$I_3 - (1,2)$

THE GRAPH I_4

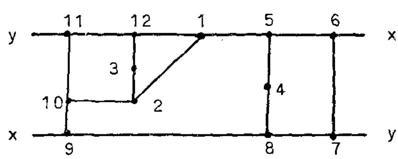


Transitivity Classes

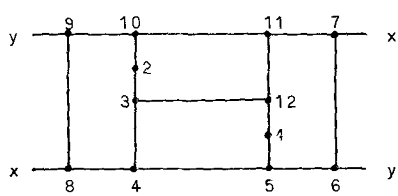
(10, 11)	(1, 2)	(3, 4)
(4, 5)	(6, 7)	(9, 10)
	(3, 12)	(1, 5)
	(8, 9)	(7, 11)
	(6, 9)	(4, 8)
	(1, 12)	(2, 10)
	(7, 8)	(5, 6)
	(2, 3)	(11, 12)

Automorphisms

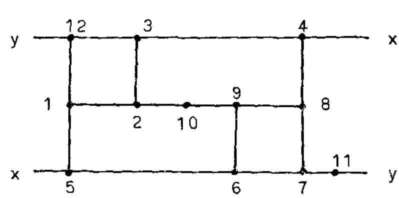
1	2	3	4	5	6	7	8	9	10	11	12
7	6	9	10	11	12	3	2	1	5	4	8
3	2	1	5	4	8	7	6	9	10	11	12
2	1	12	11	10	9	8	7	6	5	4	3



$I_4 - (3,4)$

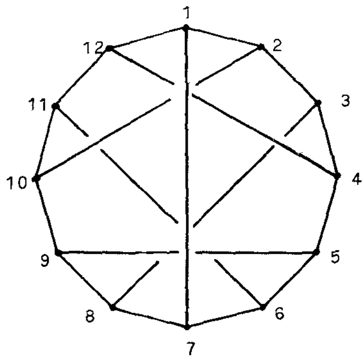


$I_4 - (2,1)$



$I_4 - (10,11)$

THE GRAPH I_5

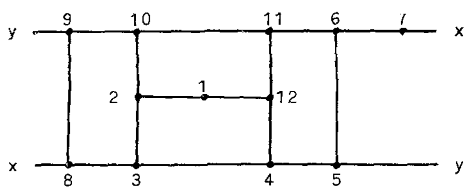


Transitivity Classes

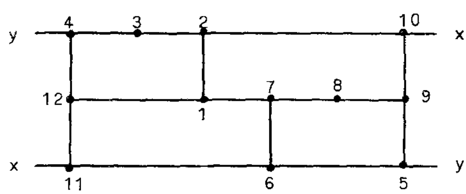
(1, 7)	(3, 8)
(3, 4)	(1, 12)
(10, 11)	(1, 2)
(9, 10)	(2, 3)
(4, 5)	(11, 12)
(6, 7)	(8, 9)
(7, 8)	(5, 6)
(4, 12)	(6, 11)
(2, 10)	(5, 9)

Automorphisms

1	2	3	4	5	6	7	8	9	10	11	12
3	2	1	7	6	5	4	12	11	10	9	8
1	12	11	10	9	8	7	6	5	4	3	2

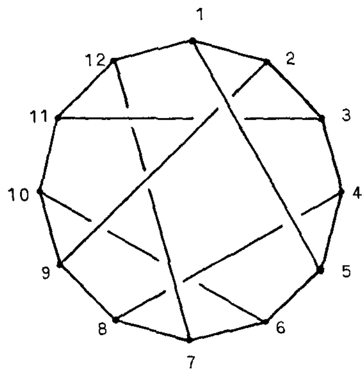


$I_5 - (1,7)$



$I_5 - (3,8)$

THE GRAPH I_6

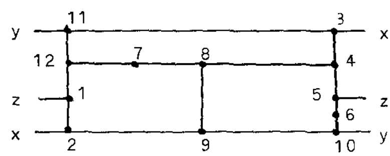


Transitivity Classes

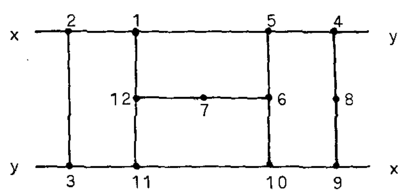
(6, 7)	(7, 8)	(4, 8)
(2, 3)	(1, 2)	(8, 9)
	(5, 6)	(9, 10)
	(3, 4)	(10, 11)
	(2, 9)	(11, 12)
	(6, 10)	(12, 1)
	(3, 11)	(1, 5)
	(7, 12)	(4, 5)

Automorphisms

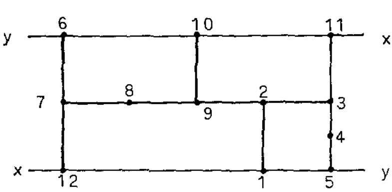
1	2	3	4	5	6	7	8	9	10	11	12
5	6	7	8	4	3	2	9	10	11	12	1



$I_6 - (6,7)$



$I_6 - (7,8)$



$I_6 - (4,8)$

9. Outline of the Proof

An *exceptional* graph is a cubic graph without extra θ, irreducibly non-representable on the projective plane which is neither I_5 nor I_6. The main result of this paper is that there are no exceptional graphs. The difference between the proof of this theorem and our previous finiteness proof is analogous to the difference between our finiteness argument in Theorem 4.3 and Kuratowski's original proof. We will start with $K_{3,3}$, which must be a subgraph of any exceptional graph and proceed to construct larger subgraphs. At each stage, large numbers of subgraphs will be rejected either because they contain an extra θ or because they contain a subgraph which is isomorphic to I_5 or I_6. Our constructions will proceed in two stages. The first is clearly motivated by the reduced representations of I_5 and I_6, namely, we will show that the graph $I_5 - (3, 8)$ which is isomorphic to $I_6 - (4, 8)$ must be a subgraph of any cubic graph without extra θ irreducibly non-representable on the projective plane. The next stage is unesthetic. We must proceed to larger graphs in order to show that they cannot be subgraphs of exceptional graphs. In no instance will a computation be left to the reader! The reader is requested to repeat the steps given and it is expected that he will require much blank paper. That we have chosen this method of proof necessarily means that it is very difficult to motivate the constructions. However, the diligent reader will be able to develop some insight.

At each stage in our proof we will consider some subgraph G' which must appear in our graph. We will then consider *all* graphs formed from G' by selecting two points and adding an edge between those points (which now become nodes of degree three). Each one of these graphs will be tested to see whether or not it can be a subgraph. The number of such possibilities is reduced considerably by the following lemma:

Lemma 9.1. *Let G be a cubic graph without extra θ irreducibly non-representable on the projective plane. Let G' be any cubic nonplanar proper subgraph of G. Then there is a subgraph G'' of G which is homeomorphic to G' such that there is a path in $G - G''$ between two non-adjacent sides of G''. (Note that G'' may be G' itself.)*

Proof. By Lemma 8.1.1, the only disconnected cubic graph irreducibly non-representable on the projective plane is with extra θ. Thus G is connected, and some side s of G' has a point n' which is a node of degree three in G, but not of degree three in G'. In the case in which there is a path in $G - G'$ from s to a side non-adjacent to s, we need go no further. Assume then that there is a path p in $G - G'$ from n' to a point n'' on a

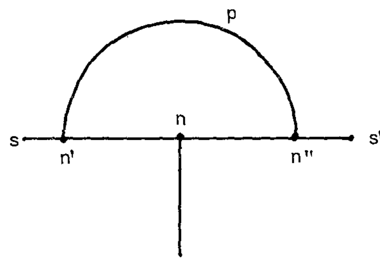


FIGURE 9.1

side s' which is adjacent to s at the node n (see Figure 9.1). From Lemma 3.2, we know that there is such a path or else G would be 1-connected. From Lemma 3.3, we conclude that $t = p + (n', n) + (n, n'')$ cannot be a 3-circuit in G. Consider the case in which there is a node m of G on (n', n). Then, the graph $G'' = G' - (n', n) + p$ is homeomorphic to G' and (n', n) is a path in $G - G''$ between two sides of G'' which are adjacent at n''. Thus, we need only consider the case in which p has a node m of G. Then there is a path q in $G - G' - p$ from m to $G' + p$. But the removal of n and the three concurrent sides of G' leaves a Th since G' is non-planar. Thus, q cannot be a path from m to either p or to one of the three sides of G' which are concurrent at n. For, if it were, there would be a θ disjoint from a Th, which is contrary to hypothesis. The only remaining possibility is that q is a path from m to a side s'' which is adjacent to s but not to s'. In this last case, $p' = (n'', m) + q$ is a path in $G - G'$ between s' and s'' which are non-adjacent sides of G' and the lemma is proved.

Remark. The proof above would hold for any surface S if G is cubic, irreducibly non-representable on S and there are no three edges whose removal disconnect G (compare Lemma 8.1.3).

The preceding lemma reduces considerably the amount of necessary computation.

10. SUBGRAPHS OF GRAPHS WITHOUT EXTRA θ

The graph K_n consists of $2n$ nodes, 1, 2, 3,..., $2n$ which form a Hamiltonian circuit (in that order) and the node i is connected to $n + i$. We are here using a notation usually reserved for the complete n-graph so that we might have $K_3 = K_{3,3}$.

The next lemma will produce the first graph larger than $K_{3,3}$ which must be a subgraph of graphs without extra θ.

LEMMA 10.1. *Let G be a cubic graph without extra θ irreducibly non-representable on the projective plane. Then G contains a proper subgraph which is homeomorphic to K_4 (See figure 10.1).*

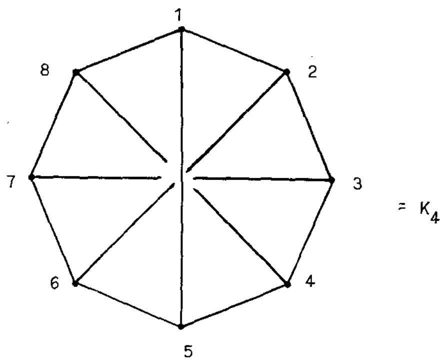


FIGURE 10.1

Proof. G must contain a proper subgraph homeomorphic to $K_{3,3}$. Thus the conditions of Lemma 9.1 are satisfied. But, for any pair of non-adjacent edges of $K_{3,3}$, there is an automorphism that will take that pair into any other pair of non-adjacent edges (see Lemma 8.1.7). Thus, adding an edge between any pair of points, one each on two non-adjacent edges of $K_{3,3}$, results in K_4. From Lemma 9.1, we conclude that G must have a subgraph homeomorphic to K_4.

LEMMA 10.2. *There is a K_n such that no cubic graph irreducibly non-representable on the projective plane has a subgraph homeomorphic to K_n.*

Proof. We will consider this a corollary to Theorem 7.14 and take, for example, K_{433}.

The edges of K_4 will divide into two classes, the *rim* which consists of edges in the Hamiltonian circuit and the *diagonals* which consist of all other edges. There is an automorphism (a rotation of Figure 10.1) which will transform any edge of the rim into any other edge of the rim, or alternately any diagonal into any other diagonal. Thus, when we are considering connections between pairs of edges of K_4, we need only consider one edge of the rim (and connections to all other edges) and one edge of the diagonal (and connections to all other edges).

LEMMA 10.3. *Let G be a cubic graph without extra θ irreducibly non-representable on the projective plane. Then G must contain a proper subgraph homeomorphic to P (Figure 10.3.1), Q (Figure 10.3.2), or R (Figure 10.3.3).*

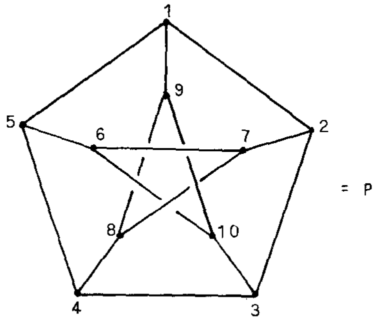
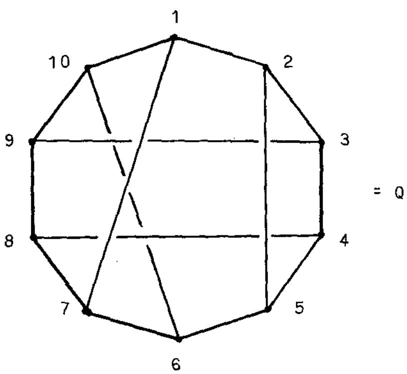


FIGURE 10.3.1



Transitivity Classes

(3, 4)	(2, 5)	(6, 7)	(6, 10)	(1, 2)	(2, 3)
	(8, 9)	(1, 10)	(1, 7)	(9, 10)	(3, 9)
				(5, 6)	(4, 8)
				(7, 8)	(4, 5)

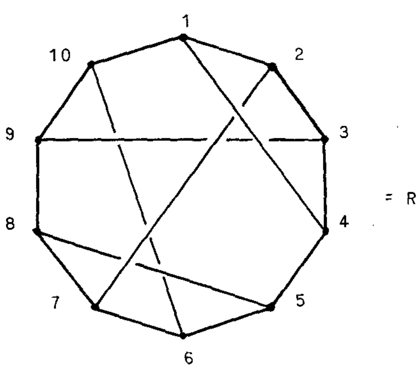
Automorphisms

1	2	3	4	5	6	7	8	9	10
10	9	3	4	8	7	6	5	2	1
6	5	4	3	2	1	10	9	8	7

FIGURE 10.3.2

Automorphisms

1	2	3	4	5	6	7	8	9	10
1	4	3	2	7	6	5	8	9	10
3	2	1	4	5	8	7	6	10	9
8	7	6	5	4	3	2	1	10	9



Transitivity Classes

(9, 10)	(4, 5)	(3, 9)	(1, 2)
	(2, 7)	(8, 9)	(2, 3)
		(1, 10)	(3, 4)
		(6, 10)	(1, 4)
			(5, 6)
			(5, 8)
			(7, 8)
			(6, 7)

FIGURE 10.3.3

Proof. From Lemma 10.1, we will consider all connections between pairs of points of K_4. From the remarks above, we will assume that one point is either on (1, 8), an edge of the rim, or on (1,5), an edge of the diagonal (see Figure 10.1). From Lemma 9.1, we consider only connections to non-adjacent edges. If we first consider all possible connections to (1, 8), then we need consider only connections from a diagonal to (1, 5), since connections to an edge of the rim will already have been considered. The entries in Table 10.3.1 give all the possibilities. In reading the table,

TABLE 10.3.1

	(1, 8)	(1, 5)
(2, 3)	Q_1	
(3, 4)	Q_2	
(3, 7)	R	P
(4, 5)	K_5	
(4, 8)		Q_3

let the point y be on the edge in the row heading and let the point x be on the edge in the columnar entry. In keeping with our promise to leave no computation to the reader, Table 10.3.2 gives the renaming conventions so that the reader need only check the figures. For example, the entry for the connection between the point x which is on (1, 8) and the point y which is on (2, 3) is "Q_1" (in Table 10.3.1). This means that the resulting graph is homeomorphic to Q and the homeomorphism is given by the corresponding renaming convention in Table 10.3.2.

TABLE 10.3.2

Renaming Conventions

	1	2	3	4	5	6	7	8	9	10
Q_1	8	x	1	2	y	3	7	6	5	4
Q_2	y	3	2	6	7	8	4	5	1	x
Q_3	6	5	x	y	4	3	7	8	1	2
P	1	2	3	y	x	5	6	7	8	4
R	2	6	5	1	x	y	7	8	4	3

The first entry in that table informs us that the mapping indicated by $8 \to 1$, $x \to 2$, $1 \to 3$, etc. is a homeomorphism to Q as defined by Figure 10.3.2. This homeomorphism is exhibited in Figure 10.3.4. The reader is requested to draw similar figures for each renaming. In

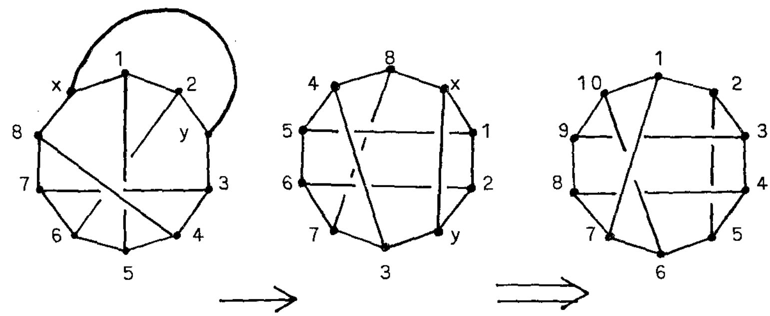


FIGURE 10.3.4

Figure 10.3.4, the ⇒ indicates renaming to conform with the definition of Q in Figure 10.3.2. The one entry in Table 10.3.1 which must be considered further is that which produces a K_5. If we further consider all connections within a K_5, any which does not produce a K_6 may be related back to one of our previous connections within K_4 (by removing a diagonal). But the same argument holds for connections between edges of a subgraph homeomorphic to a K_n such that the connections do not produce a K_{n+1}, namely, if a connection between a point of K_n to another point of K_n does not produce a K_{n+1}, then by removing diagonals we have one of our previous cases of connections within K_4. But Lemma 10.2 precludes the possibility of building larger and larger K_n.

Figure 10.3.5 shows that each K_n is representable on the projective plane. Thus, if K_n appears as a subgraph of an irreducible graph, it is a

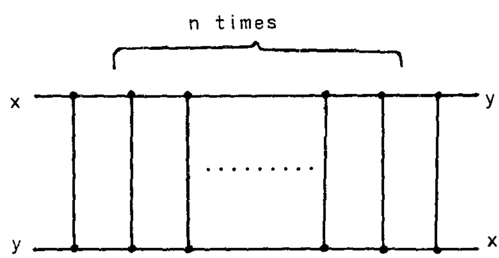


FIGURE 10.3.5

proper subgraph and the conditions of Lemma 9.1 hold. Thus, the other possibilities for connections in K_4 are all those that need be considered. In order to show that P, Q, and R are proper subgraphs, we need only give representations on the projective plane for these graphs. But the three reduced graphs of I_6 are precisely P, Q, and R. This can be seen in Table 10.3.3, which gives the renaming of the three graphs. The renaming of the nodes gives the explicit homeomorphism.

TABLE 10.3.3

Renaming Conventions

	1	2	3	4	5	6	7	8	9	10
$I_6 - (6, 7) = P$	11	3	4	8	12	1	2	9	10	5
$I_6 - (7, 8) = Q$	2	1	12	6	5	4	9	10	11	3
$I_6 - (4, 8) = R$	7	6	10	9	2	1	5	3	11	12

Remark. There are six parts to the proof above. These parts will be repeated in succeeding proofs:

(1) A subgraph is found which must appear as a proper subgraph.

(2) Automorphisms are found so that only one edge of each transitivity class is considered for the first of the pair of edges to be connected.

(3) Only non-adjacent pairs of edges are considered (Lemma 9.1).

(4) Renaming conventions are given to enable the reader to verify the homeomorphism to the new subgraphs.

(5) Those exceptional cases are listed.

(6) Finally, those new graphs are given a representation on the projective plane in order to show that they must be proper subgraphs of irreducible graphs.

It should be noted that Lemma 10.3 is motivated by the reduced representations of I_5 and I_6.

The graph P is the *Petersen* graph. Its automorphisms can be seen at a glance if it is given the following representation:

> Consider the ten unordered sets of two distinct integers $\{a, b\}$ where a and b are selected as 1, 2, 3, 4, or 5. These ten sets will represent the nodes of the Petersen graph. Two such nodes, A and B will be adjacent if and only if, as sets, $A \cap B = \varnothing$ (see Figure 10.4).

The representation above shows that there is only one transitivity class of edges. Thus, we will select one edge and consider only connections to that one edge. With respect to that one edge, all other edges may be divided into two classes. Any edge is either at distance one or at distance two from the selected edge. Two edges in a graph are *at distance n* if the shortest path between them has n edges. Thus, with respect to a given edge of P there are four edges at distance zero, eight edges at distance one, and two at distance two. These distances are given in Table 10.4.

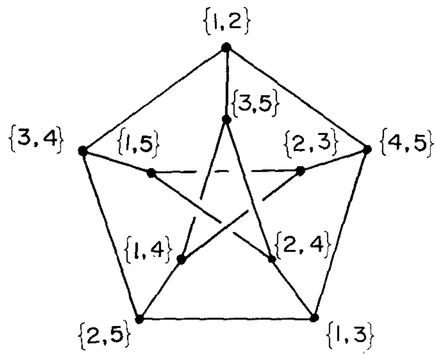


FIGURE 10.4

The table also shows how to permute edges within each class. For example, the edge $\{4, 2\} - \{1, 3\}$ may be permuted with the edge $\{4, 5\} - \{2, 3\}$ by the following permutation of the integers in the representation of the end nodes:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 1 & 4 & 2 \end{pmatrix}.$$

We note that the permutation in this example leaves the edge $\{3, 5\} - \{1, 2\}$ fixed although it does permute the end nodes. Thus, the representation of P given in Figure 10.4 and entries in Table 10.4 provide a demonstration of:

LEMMA 10.4. *Let e be an edge of P, and let f and g be edges at distance n from e. Then there is an automorphism of P which leaves e fixed but permutes f and g. (The end nodes of e might be permuted, however.)*

TABLE 10.4

Distance from $\{3, 5\} - \{1, 2\}$

Distance =	2	0	1
	$\{2, 5\} - \{3, 1\}$	$\{3, 5\} - \{1, 4\}$	$\{4, 2\} - \{1, 3\}$
	$\{2, 3\} - \{1, 5\}$	$\{3, 5\} - \{4, 2\}$	$\{4, 2\} - \{1, 5\}$
		$\{1, 2\} - \{3, 4\}$	$\{1, 4\} - \{2, 5\}$
		$\{1, 2\} - \{4, 5\}$	$\{1, 4\} - \{2, 3\}$
			$\{3, 4\} - \{1, 5\}$
			$\{3, 4\} - \{2, 5\}$
			$\{4, 5\} - \{1, 3\}$
			$\{4, 5\} - \{2, 3\}$

LEMMA 10.5. *Let G be a cubic graph without extra θ irreducibly non-representable on the projective plane. If G contains a subgraph homeomorphic to P, then it also contains a subgraph homeomorphic to R.*

Proof. We will now use the representation of Figure 10.3.1 for the Petersen graph. Since there is only one transitivity class, we will consider only connections from the edge (1, 2). Since P must be a proper subgraph of G, the conditions of Lemma 9.1 are satisfied and we need only consider connections to edges at distance one or two from (1, 2). From Lemma 10.4, we need only consider connections from (1, 2) to any one selected edge, say, (9, 10) at distance one, and to any one selected edge, say, (6, 10) at distance two. Now, the connection between a point x on (1, 2) and a point y on (6, 10) yields I_6 with the renaming:

1	2	3	4	5	6	7	8	9	10	11	12
7	8	4	5	6	y	x	1	9	10	3	2

If y is on (9, 10), then the connection yields an R when the edge (4, 5) is removed; renamed as:

1	2	3	4	5	6	7	8	9	10
7	8	3	2	x	1	9	y	10	6

Remark. The lemma above could have been stated with "Q" for "R" but we are here only interested in R.

THEOREM 10.6. *Let G be a cubic graph without extra θ irreducibly non-representable on the projective plane. Then G has a proper subgraph homeomorphic to R.*

Proof. By Lemmas 10.3 and 10.5, we need only consider the case in which G contains a Q. From Theorem 7.4, we conclude that at least one of the edges (2, 3), (3, 4), (4, 5), (5, 2), (3, 9), (4, 8), (8, 9) of Q must have an extra point which is a node of G. Otherwise, those edges would form an outside graph with a θ connected to the rest of G by the four edges (1, 2), (5, 6), (9, 10), and (6, 7). But, by the automorphisms shown, we need only consider connections from only three of those seven edges (one of which must have an extra point). These three edges are the column headings in Table 10.3.2.1. The entries in that table give all possibilities for connections to non-adjacent edges (which are the row headings). The point "x" is on the edge indicated by the column heading and the point "y" is on the edge indicated by the row heading. The columns have been filled in from left to right so that in any column there are no connections indicated to edges that are in the same transitivity class as an edge in a column heading to the left. That type of connection will have already

TABLE 10.3.2.1

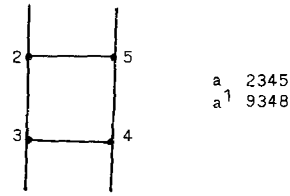
	(3, 4)	(2, 3)	(2, 5)
(4, 8)		a^1	
(8, 9)	a^1	a^1	
(6, 7)	$I_6{}^1$	$R_{(2,5)}$	$R^1_{(3,4)}$
(7, 8)	a^1	a^1	$R_{(4,8)}$
(6, 10)	$I_5{}^1$	$R^2_{(3,4)}$	$R_{(1,10)}$
(1, 7)	$I_5{}^2$	$R^3_{(3,4)}$	
(5, 6)	a	a	
(9, 10)	a^1	a^1	
(1, 10)	$I_6{}^2$	$R_{(6,7)}$	
(4, 5)		a	
(2, 5)	a		

been considered. Thus Table 10.3.2.1 has no entry for a connection between (2, 5) and (4, 8) since the edge (4, 8) is in the same transitivity class as (2, 3) and all connections to (2, 3) had been entered previously. This will eliminate many cases. The subscripts show which edges are to be removed in the homeomorphism to R and the superscripts (for the R's) indicate what renaming is necessary for Table 10.3.2.2 that are the renaming conventions showing the homeomorphisms to I_6, I_5, or R. The entries with a's indicate that there is a θ disjoint from a Th. Figure 10.3.2.1 exhibits two 4-circuits that are disjoint from a Th in Q (refer to Figure 10.3.2). Thus when a connection (e.g., (3, 4) to (5, 6)) together with a 4-circuit form a θ, it must be a θ disjoint from a Th. The tables give all possibilities with the explicit homeomorphisms and the theorem is proved.

TABLE 10.3.2.2

Renaming Conventions

	1	2	3	4	5	6	7	8	9	10	11	12
$R_{(6,7)}$	y	1	2	x	3	9	8	4	5	10		
$R_{(4,8)}$	10	6	7	1	2	3	5	x	y	9		
$R_{(1,10)}$	3	9	8	4	5	x	y	6	7	2		
$R_{(2,5)}$	1	7	y	x	3	9	8	4	6	10		
$R^1_{(3,4)}$	y	6	5	x	2	1	10	9	8	7		
$R^2_{(3,4)}$	x	9	10	y	6	5	8	7	1	2		
$R^3_{(3,4)}$	x	2	1	y	7	8	5	6	10	9		
$I_5{}^1$	x	3	9	8	7	6	y	10	1	2	5	4
$I_5{}^2$	x	3	2	5	6	7	y	1	10	9	8	4
$I_6{}^1$	2	1	10	6	5	4	x	y	7	8	9	3
$I_6{}^2$	5	6	7	1	2	3	x	y	10	9	8	4



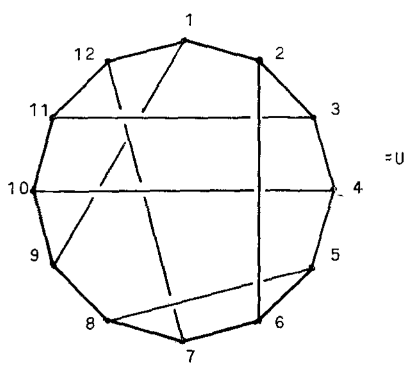
Subgraphs Disjoint from a Th

FIGURE 10.3.2.1

11. THE EXCEPTIONAL GRAPH

To complete the proof of the main result of this paper, we will assume the existence of an exceptional graph and, by going to larger subgraphs, adduce a contradiction.

LEMMA 11.1. *Let G be an exceptional graph. Then G must contain as a proper subgraph one of the five following cubic graphs: U (Figure* 11.1.1), *V (Figure* 11.1.2), *W (Figure* 11.1.3), *X (Figure* 11.1.4), *or Y (Figure* 11.1.5).



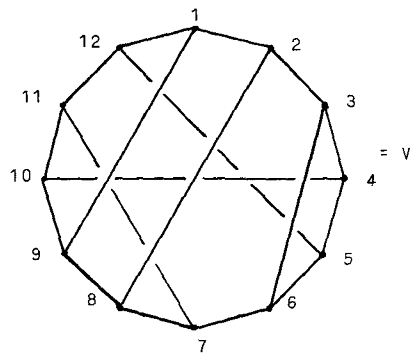
Automorphisms

1	2	3	4	5	6	7	8	9	10	11	12
1	2	6	5	4	3	11	10	9	8	7	12
1	9	8	5	4	10	11	3	2	6	7	12

Transitivity Classes

(4, 5)	(1, 12)	(1, 2)	(7, 12)	(3, 4)	(3, 11)	(2, 3)
		(1, 9)	(11, 12)	(4, 10)	(10, 11)	(2, 6)
				(5, 6)	(6, 7)	(9, 10)
				(5, 8)	(7, 8)	(8, 9)

FIGURE 11.1.1



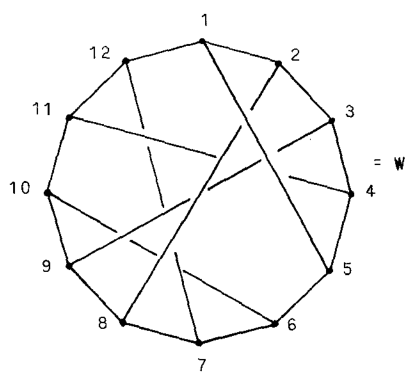
Automorphisms

1	2	3	4	5	6	7	8	9	10	11	12
4	3	2	1	9	8	7	6	5	12	11	10

Transitivity Classes

(2, 3)	(7, 11)	(2, 8)	(1, 2)	(4, 5)	(5, 6)	(6, 7)	(9, 10)	(10, 11)	(1, 12)
		(3, 6)	(3, 4)	(1, 9)	(8, 9)	(7, 8)	(5, 12)	(11, 12)	(10, 4)

FIGURE 11.1.2



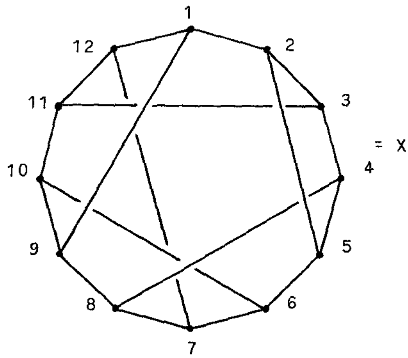
Automorphisms

1	2	3	4	5	6	7	8	9	10	11	12
4	3	9	10	11	12	1	2	8	7	6	5
1	2	8	7	12	11	4	3	9	10	6	5

Transitivity Classes

(5, 6)	(1, 2)	(2, 3)	(4, 5)
(11, 12)	(3, 4)	(3, 9)	(10, 11)
	(9, 10)	(8, 9)	(6, 7)
	(7, 8)	(2, 8)	(1, 12)
			(1, 5)
			(7, 12)
			(10, 6)
			(4, 11)

FIGURE 11.1.3



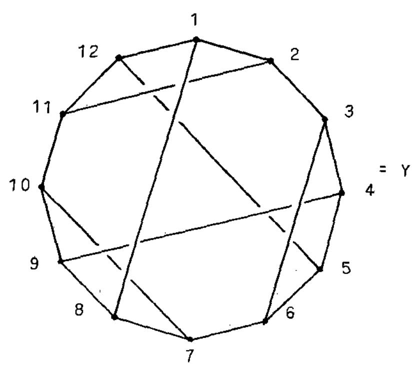
Automorphisms

1	2	3	4	5	6	7	8	9	10	11	12
6	5	4	3	2	1	12	11	10	9	8	7
8	4	5	2	3	11	12	1	9	10	6	7

Transitivity Classes

(7, 12)	(9, 10)	(2, 3)	(3, 4)	(1, 2)	(6, 7)	(8, 9)
		(4, 5)	(2, 5)	(5, 6)	(1, 12)	(10, 11)
				(4, 8)	(11, 12)	(1, 9)
				(3, 11)	(7, 8)	(6, 10)

FIGURE 11.1.4



Automorphisms

1	2	3	4	5	6	7	8	9	10	11	12
12	1	8	7	10	9	4	5	6	3	2	11
5	6	7	8	9	10	11	12	1	2	3	4
12	11	10	9	8	7	6	5	4	3	2	1

Transitivity Classes

(1, 8)	(1, 2)
(4, 9)	(12, 1)
(5, 12)	(5, 6)
(2, 3)	(11, 12)
(7, 6)	(2, 11)
(11, 10)	(3, 4)
	(7, 8)
	(9, 10)
	(8, 9)
	(4, 5)
	(7, 10)
	(3, 6)

FIGURE 11.1.5

Proof. We know from Theorem 10.6 that R must be proper subgraph of G. Thus, the conditions of Lemma 9.1 hold and Table 11.1.1 is constructed giving all possibilities for connections within R.

TABLE 11.1.1

	(1, 2)	(3, 9)	(2, 7)
(1, 2)			
(2, 3)			
(3, 4)	a		
(1, 4)			
(5, 6)	I_5		
(5, 8)	I_6		
(7, 8)	X_1		
(6, 7)	W		
(3, 9)	a		
(8, 9)	X_2		
(1, 10)		a	
(6, 10)	V_1	Y	
(4, 5)	a	a	a
(2, 7)		a	
(9, 10)	V_2		U

Recall again that the columns in that table are filled in from left to right so that there are no entries for connections to an edge in a transitivity class that has been considered previously (refer to Figure 10.3.3). The entry indicated by the letter a signifies that the connection together with the 4-circuit, 1234, froms a θ disjoint from a Th. Note that half-edges at the nodes of the 4-circuit are included in the configuration making up the θ. Thus, a connection from a point x on (2, 7) to a point y on (4, 5) makes the following θ disjoint from a Th: the 4-circuit, 1234, the connection (x, y), the half-edge $(x, 2)$ of (2, 7), and the half-edge $(y, 4)$ of (4, 5). The Th (which is disjoint from the θ) is formed by the nodes 6, 7, 8, 9, 10 and the edges between them. First, the reader should see that the 4-circuit is indeed disjoint from a Th. Then each entry, a for a connection between (b, c) and (d, e) is verified if and only if at least one of b, c is a node of the 4-circuit and at least one of d, e is a node of the 4-circuit. The subscripts in Table 11.1.1 indicate which explicit homeomorphism is used for the renaming conventions in Table 11.1.2.

The proof is completed by giving representations of the graphs on the projective plane. For, if a graph is a subgraph of a non-representable graph and the subgraph is itself representable, then the subgraph must be a proper subgraph. The representations are given in Figure 11.1.6.

TABLE 11.1.2

Renaming Conventions

	1	2	3	4	5	6	7	8	9	10	11	12
Y	10	y	6	7	8	5	4	1	2	3	x	9
U	y	9	3	4	5	8	7	6	10	1	2	x
V_1	10	y	6	5	8	7	2	x	1	4	3	9
V_2	y	10	6	7	8	5	4	1	x	2	3	9
W	1	x	2	3	4	5	6	y	7	8	9	10
X_1	3	2	x	y	7	6	5	8	9	10	1	4
X_2	10	6	5	8	7	2	x	y	9	3	4	1
I_5	4	1	x	2	7	8	5	y	6	10	9	3
I_6	3	4	1	x	2	7	8	y	5	6	10	9

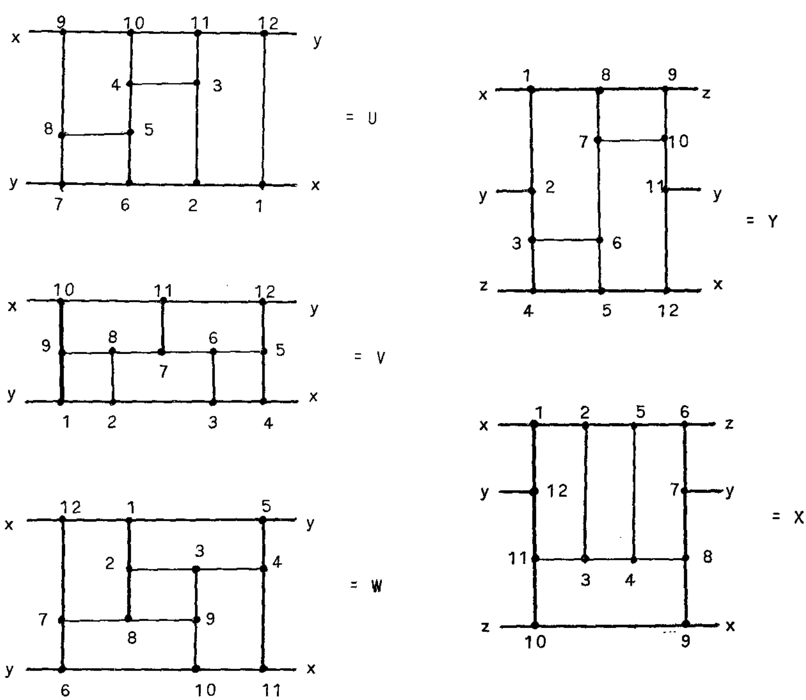


FIGURE 11.1.6

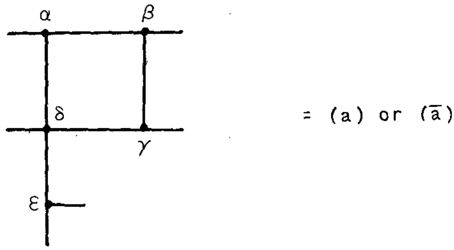


FIGURE 11.1.7

In what follows, we will be concerned with two important subgraphs, each of which is a five node configuration. We will refer to these as the *a* (Figure 11.1.7) or *b* (Figure 11.1.8) configuration. Let G be a cubic graph irreducibly non-representable on the projective plane and let G' be a non-planar proper subgraph of G. Then if G' has an *a* or *b* configuration which is disjoint from a Th in G', then a connection between non-adjacent sides of G' (in accordance with Lemma 9.1) together with either the *a* or *b* configuration cannot form a θ.

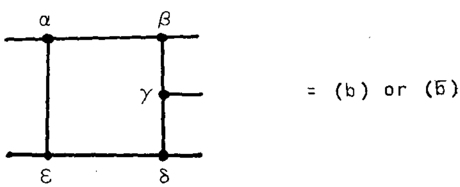


FIGURE 11.1.8

ALGORITHM 11.1.1. Let α, β, γ, δ, ϵ of G' form an *a* or *b* configuration (see Figures 11.1.7 and 11.1.8) which is disjoint from a Th in G'. Then a connection between two non-adjacent sides, (c, d) and (e, f), of G' form a θ (which is disjoint from the Th in G') if at least one of c, d and at least one of e, f is of the five nodes α, β, γ, δ, ϵ.

We will henceforth indicate an *a* or *b* configuration by those letters and a superscript. This superscripted letter will be the row heading of a table naming α, β, γ, δ, and ϵ. When the superscripted letter appears as an entry in a table which gives the result of a connection between edges, it indicates that the connection and the configuration referred to form a θ disjoint from a Th.

The reader must verify that the nodes named for α, β, γ, δ, and ϵ do form the configuration disjoint from a Th. In several (but not all) cases the configurations indicated by different superscripts are images of each other by some automorphism. After the configurations are verified, Algorithm 11.1.1 must be used to verify that each entry for connections between edges is correct. No mental effort is required on the part of the reader!

Now, the configurations above may be used in a complementary way; the configuration may be disjoint from a θ in G' and the configuration together with the connection between non-adjacent sides of G' form a Th disjoint from the θ in G'. When the configurations are used in this complementary manner, we will refer to them as $\bar{a}$ and $\bar{b}$ configurations even though they are identical to the a and b configurations in Figures 11.1.7 and 11.1.8.

ALGORITHM 11.1.2. Let α, β, γ, δ, ϵ of G' form an $\bar{a}$ configuration disjoint from a θ in G'. Then a connection between two non-adjacent sides (c, d) and (e, f) of G' form Th disjoint from a θ in G' if:

(i) c, d, e, and f are four nodes of α, β, γ, δ, ϵ which include δ and ϵ (see Figure 11.1.7);

(ii) c, d, e, and f comprise three of α, β, γ, δ, ϵ but not $(\epsilon, \delta, \gamma)$ or $(\epsilon, \delta, \alpha)$;

(iii) one of c, d say g and one of e, f say h is of α, β, γ, δ, ϵ but not ϵ and α or ϵ and γ nor is (g, h) an edge of the $\bar{a}$ configuration (see Figure 11.1.7).

ALGORITHM 11.1.3. Let $\alpha, \beta, \gamma, \delta, \epsilon$ of G' form a $\bar{b}$ configuration disjoint from a θ in G'. Then a connection between two non-adjacent sides (c, d) and (e, f) of G' forms a Th disjoint from a θ in G' if:

(i) $c, d, e,$ and f comprise exactly three of $\alpha, \beta, \gamma, \delta, \epsilon$ (see Figure 11.1.8);

(ii) one of c, d say g and one of e, f, say h is in α, β, γ, δ, ϵ but (g, h) is not an edge of the $\bar{b}$ configurations (see Figure 11.1.8).

The reader will go through the verification steps for the $\bar{a}$ and $\bar{b}$ configurations in exactly the same way as for the a and b configurations.

Remark. The a and $\bar{a}$ (or b and $\bar{b}$) configurations are identical. Their uses, however, are complementary: one forms a θ disjoint from a Th and the other forms a Th disjoint from a θ. Note also that the verification algorithm for the a and b (but not for $\bar{a}$ and $\bar{b}$) configuration is identical.

We will treat each of our five proper subgraphs (one of which must appear) separately. We will develop a table for each giving all possible connections between edges. Renaming conventions will be given which exhibit explicit homomorphisms either to graphs considered previously or to a new subgraph. When an edge is used as a subscript for a graph, as before, it indicates that the edge must be removed to obtain the indicated graph. Superscripts for graphs indicate which homeomorphism is to be used for the renaming conventions which give the explicit homeomorphism.

LEMMA 11.2. *Let G be an exceptional graph which has a proper subgraph homeomorphic to U. Then G has a proper subgraph homeomorphic to E (Figure* 11.2).

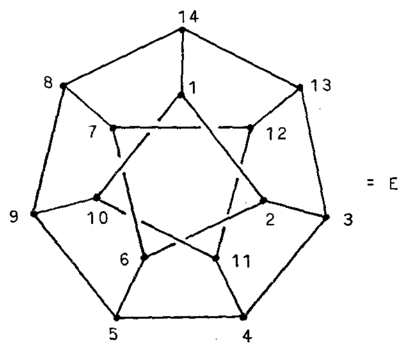


FIGURE 11.2

TABLE 11.2

	(2, 3)	(6, 7)	(3, 4)	(7, 12)	(1, 2)	(1, 12)
(2, 3)						
(2, 6)						
(9, 10)	a					
(8, 9)	$\bar{a}$					
(3, 11)		b				
(10, 11)	a	E				
(6, 7)	b					
(7, 8)	$\bar{b}$					
(3, 4)		b				
(4, 10)	b	b				
(5, 6)	b		b			
(5, 8)	b	b	b			
(7, 12)	$\bar{b}$		a			
(11, 12)	a	$\bar{b}$	a			
(1, 2)		b	b	$\bar{b}^1$		
(1, 9)	$\bar{a}$	$\bar{b}$	$\bar{a}$	$\bar{b}$		
(1, 12)	a	$\bar{b}^1$	a			
(4, 5)	b	b		a	b	a

Proof. Table 11.2 gives all relevant entries. Table 11.2.1 exhibits the configurations necessary for the entries. Table 11.2.2 gives the explicit homeomorphism. The last step is to give E a representation on the projective plane. This is done in Figure 11.2.2 and the lemma is proved.

LEMMA 11.3. *Let G be an exceptional graph which has a proper subgraph homeomorphic to V. Then G has a proper subgraph homeomorphic to E.*

TABLE 11.2.1

		α	β	γ	δ	ϵ
Subgraphs disjoint from a Th	a	10	4	3	11	12
	b	2	3	4	5	6
	$\bar{a}$	4	3	11	10	9
Subgraphs disjoint from a θ	$\bar{b}$	1	2	6	7	12
	$\bar{b}^1$	1	9	8	7	12

TABLE 11.2.2

Renaming Conventions

	1	2	3	4	5	6	7	8	9	10	11	12	13	14
E	2	1	12	7	8	9	10	4	5	6	x	y	11	3

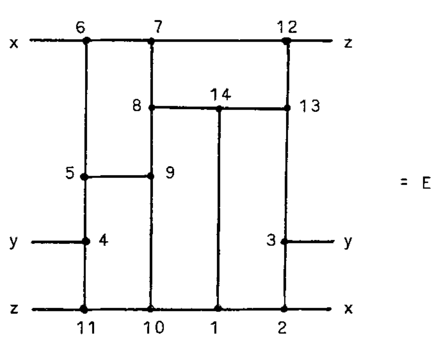


FIGURE 11.2.2

Proof. Table 11.3 gives all relevant entries. Table 11.3.1 gives the a and b configurations. The renaming conventions giving the explicit homeomorphism are in Table 11.3.2. We note that one possible connection in V will yield a subgraph homeomorphic to U. In that case, we apply Lemma 11.2 and the lemma is proved.

TABLE 11.3

	(1, 2)	(4, 5)	(5, 6)	(6, 7)	(9, 10)	(10, 11)	(1, 12)	(2, 8)	(7, 11)
(1, 2)									
(3, 4)	a								
(4, 5)	a								
(1, 9)		E							
(5, 6)	a								
(8, 9)	a^1	$I_{6(2,8)}$	$I_{6(1,9)}$						
(6, 7)	a	a							
(7, 8)	a^1	b	b						
(9, 10)	a^1	a^2	a^2	a^2					
(5, 12)	a			a	a^2				
(10, 11)	$\bar{b}$	a^2	b	b					
(11, 12)	$\bar{b}$	b	b	b	b^1				
(1, 12)		b	b	b	a^1	b			
(10, 4)	a		a	a			$U_{(2,8)}$		
(2, 8)		a	a	a	b^1	b^1	a^1		
(3, 6)	a	a			a^2	a^2	a^1	a	
(7, 11)	b^2	b	b		b^1		b	b^1	
(2, 3)		a	a	a	a^2	a^2	a^1		b^2

TABLE 11.3.1

		α	β	γ	δ	ϵ
Subgraphs disjoint from a Th	a	6	5	4	3	2
	a^1	8	9	1	2	3
	a^2	5	6	3	4	10
	b	5	6	7	11	12
	b^1	9	8	7	11	10
	b^2	2	3	6	7	8
Subgraphs disjoint from a θ	$\bar{b}$	1	9	10	11	12

TABLE 11.3.2
Renaming Conventions

	1	2	3	4	5	6	7	8	9	10	11	12	13	14
E	1	2	8	7	6	3	4	x	5	12	11	10	9	y
$U_{(2,8)}$	x	12	11	7	6	5	4	3	1	9	10	y		
$I_{6(2,8)}$	3	6	5	x	4	10	9	y	7	11	12	1		
$I_{6(1,9)}$	11	7	8	y	10	4	5	x	6	3	2	12		

LEMMA 11.4. *No exceptional graph has a proper subgraph homeomorphic to W.*

Proof. Table 11.4 gives all relevant entries for connections between edges of W when we assume, contrary to hypothesis, that W is a sub-

TABLE 11.4

	(4, 5)	(2, 3)	(1, 2)	(5, 6)
(4, 5)				
(10, 11)	b^1			
(6, 7)	$\bar{b}$			
(1, 12)	b			
(1, 5)				
(7, 12)	$\bar{b}$			
(10, 6)	b^1			
(4, 11)				
(2, 3)	b			
(3, 9)	b			
(8, 9)	b^1	b^2		
(2, 8)	b			
(1, 2)	b			
(3, 4)			b	
(9, 10)	b^1	b^1	b	
(7, 8)	$\bar{b}$	b^2	b^2	
(5, 6)		b	b	
(11, 12)	b^1	b^2	b^2	$\bar{b}$

TABLE 11.4.1

		α	β	γ	δ	ϵ
	b	1	2	3	4	5
Subgraphs disjoint from a Th	b^1	4	3	9	10	11
	b^2	1	2	8	7	12
Subgraphs disjoint from a θ	$\bar{b}$	1	12	7	6	5

graph. Table 11.4.1 gives the b and $\bar{b}$ configurations. We note that in every case there is a θ disjoint from a Th and the lemma is proved.

LEMMA 11.5. *Let G be an exceptional graph which has a proper subgraph homeomorphic to X. Then G has a subgraph homeomorphic to E.*

Proof. Table 11.5 gives all relevant entries for connections between edges of X. Table 11.5.1 exhibits the a, b, and $\bar{b}$ configurations. The renaming conventions given in Table 11.5.2 exhibit the explicit homeomorphisms and the lemma is proved.

TABLE 11.5

	(8, 9)	(6, 7)	(1, 2)	(3, 4)	(2, 3)	(9, 10)
(8, 9)						
(10, 11)	$\bar{b}$					
(1, 9)						
(6, 10)	b^1					
(6, 7)	b^1					
(1, 12)	b^2	b^2				
(11, 12)	b^2	b^2				
(7, 8)						
(1, 2)	b^2	a				
(5, 6)	b^1		a			
(4, 8)		b^1	a			
(3, 11)	$\bar{b}$	a	b			
(3, 4)	b^1	b^1	b			
(2, 5)	b^1	b^1		b		
(2, 3)	$I_{6(1,9)}$	a				
(4, 5)	b^1	b^1	a		a	
(9, 10)		b^2	b^2	E	$I_{6(4,5)}$	
(7, 12)	b^1		b	b	b	$\bar{b}$

TABLE 11.5.1

		α	β	γ	δ	ϵ
	a	2	3	4	5	6
Subgraphs disjoint from a *Th*	b	1	2	3	11	12
	b^1	6	7	8	4	5
	b^2	1	12	7	8	9
Subgraphs disjoint from a θ	$\bar{b}$	1	12	11	10	9

TABLE 11.5.2

Renaming Conventions

	1	2	3	4	5	6	7	8	9	10	11	12	13	14
E	1	2	3	x	4	5	6	7	8	9	y	10	11	12
$I_{6(4,5)}$	1	12	7	8	9	y	x	3	11	10	6	2		
$I_{6(1,9)}$	5	6	7	8	4	3	y	x	10	11	12	2		

THEOREM 11.6. *Every exceptional graph has a proper subgraph homeomorphic to E.*

Proof. From Lemmas 11.1, 11.2, 11.3, 11.4, and 11.5 we need only consider the case in which Y is a proper subgraph. Table 11.6 gives all entries for connections between edges of Y. Table 11.6.1 gives the a configurations. We conclude that Y cannot be a proper subgraph and the theorem is proved.

Remark. There are several methods that could have been used to reduce the number of cases to be considered. The most obvious is the development of results similar to that of Lemma 10.4. That is, we could

TABLE 11.6

	(1, 2)	(1, 8)
(1, 2)		
(12, 1)		
(5, 6)	a	
(11, 12)	a^1	
(2, 11)		
(3, 4)	a	
(7, 8)	a^2	
(9, 10)	a^2	
(8, 9)	a^2	
(4, 5)	a	
(7, 10)	a^2	
(3, 6)	a	
(1, 8)		
(4, 9)	a	a^2
(5, 12)	a	a^1
(2, 3)		a^1
(7, 6)	a^2	a^2
(10, 11)	a^2	a^2

TABLE 11.6.1

		α	β	γ	δ	ϵ
Subgraphs disjoint from a *Th*	a	6	5	4	3	2
	a^1	2	1	12	11	10
	a^2	9	10	7	8	1

consider automorphisms that leave an edge fixed. This will be done when we consider the graph E, but we felt that the preceding five graphs should be treated in a uniform manner.

12. The Characterization of Irreducible Graphs

We note that a rotation of Figure 11.2 is an automorphism of E and thus that there are three transitivity classes of edges in E, namely, those edges in the 7-circuit, 3 4 5 9 8 14 13, those edges in the 7-circuit disjoint from the aforementioned 7-circuit, and finally the other seven edges. Now, a mirror image is also an automorphism of E. Thus, a rotation will move any edge to a selected member of its transitivity class and an additional mirror automorphism will halve the number of connections considered. For example, there is an automorphism that will keep (13, 14) fixed (although the end nodes are permuted) and permute (3, 4) and (8, 9). Of course, this is exactly analogous to Lemma 10.4 and it makes the computation shorter.

Remark. Note that, in every place in this paper in which we have made statements about transitivity classes, at no time was it necessary to prove that there was no automorphism which moved an edge of one transitivity class into an edge of another transitivity class. The reader should not accept as proved what is not necessary to this paper. We are interested here only in the fact that two edges are in the same transitivity class; we never need the fact that two edges are not in the same class. The fact there is no automorphism between the transitivity classes that we have listed is a corollary of the results which consider all possible connections between edges.

LEMMA 12.1. *No cubic graph irreducibly non-representable on the projective plane has a proper subgraph homeomorphic to E.*

Proof. Assume, contrary to hypothesis, that E is a proper subgraph. It is certainly not a subgraph of those graphs with extra θ or I_5 or I_6 since these graphs have at most 12 nodes. Thus the conditions of

Lemma 9.1 hold and we will consider all connections between non-adjacent edges of E. The connection between x on (13, 14) and y on (5, 6) produces an I_3 renamed as:

1	2	3	4	5	6	7	8	9	10
4	3	1	14	8	5	y	7	12	11

when (2, 6), (9, 10), and (13, x) are removed. From the remarks above, we only need consider those entries in Table 12.1. Table 12.1.1 gives the b configuration. In every case a θ is produced disjoint from a Th which provides the contradiction to prove the theorem.

We have just shown that I_1 to I_6 are the only irreducible graphs, for Theorem 11.6 shows that any other graph must have E as a subgraph and Lemma 12.1 shows that no irreducible graph can have such a subgraph.

LEMMA 12.2. *The six graphs I_1 to I_6 are the only cubic graphs which are irreducibly non-representable on the projective plane.*

Remark. Four of these graphs were known to Kagno [3]: I_1 in his type I (page 58), I_2 is his G_{v-3} (page 63), I_3 is his G_{iv-a-1} (page 62), and I_6 is his G_{iii-21}. Neither he nor anyone else was able to prove finiteness, let alone provide a list.

Kagno's remark that his graphs of type II were reducible (page 61)

TABLE 12.1.

	(14, 13)	(14, 1)	(1, 2)
(3, 4)	b		
(4, 5)	b		
(2, 3)	b	b	
(4, 11)	b	b	
(1, 2)	b^1		
(2, 6)	b^1	b^1	
(10, 11)	b		
(12, 13)		b^1	
(7, 12)		b^2	b^2
(6, 7)		b^2	b^2

TABLE 12.1.1

		α	β	γ	δ	ϵ
Subgraphs disjoint from a *Th*	b	3	4	11	12	13
	b^1	13	3	2	1	14
	b^2	14	13	12	7	8

led us to consider cubic graphs only, and thus provided motivation for the development of the concept of graphs without extra θ. In the 35 years between the appearence of Kagno's paper and this paper, there have been announcements (too numerous to include here) of partial or complete solutions to Kagno's problem of listing the irreducible graphs for the next simplest surfaces after the plane, but we have never been able to find either a list or a proof.

THEOREM 12.3 (Characterization Theorem). *The cubic graphs which are irreducibly non-representable on the projective plane are*:

(i) *The four graphs* (I_1 *through* I_4) *which have two disjoint non-outside subgraphs.*

(ii) *One graph* I_6 *which consists of two six-circuits and connections between them in the following order*:

$$\begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 5 & 3 & 6 & 2 & 4 \end{matrix}$$

(iii) *One graph* I_5 *which consists of a 7-circuit whose nodes are identified with the free nodes of a tree in the following order*:

$$\begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 3 & 5 & 1 & 4 & 2 & 5 \end{matrix}$$

Proof. Lemma 12.2 shows that I_1 to I_6 are the only cubic graphs which are irreducibly non-representable on the projective plane. The graphs I_1 to I_4 are, of course, with extra θ (see Lemma 8.1.5). Figure 12.3.1 shows that I_6 consists of two six-circuits. Figure 12.3.2 shows that I_5 consists of a 7-circuit and a tree.

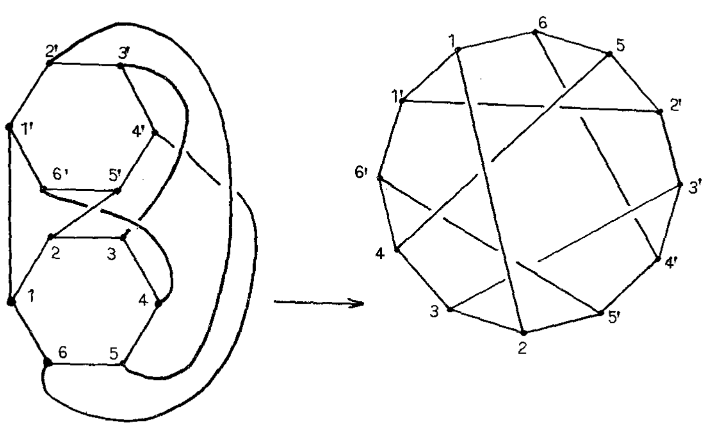


FIGURE 12.3.1

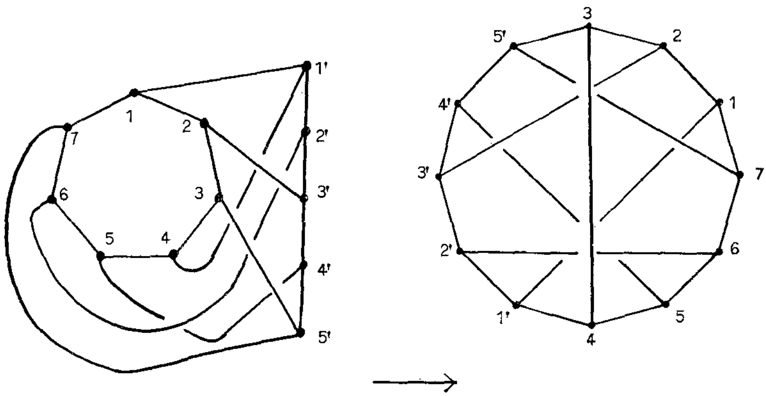


FIGURE 12.3.2

ACKNOWLEDGMENTS

The importance of the solution to this problem was impressed upon me by Professor Peter Ungar. It was he who urged me to complete this paper even though it is neither simple nor elegant.

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